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A DESCRIPTION OF THE WEIERSTRASS GAP SEQUENCE BY MEANS OF THE RIEMANN THETA FUNCTION

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Dedicated to Professor Yukio Kusunoki on his 60th birthday

ABSTRACT

We show that the vanishing of the Riemann theta function on some subvarieties of the Jacobi varieties of a compact Riemann surface determines the gap sequence at the base point of the Jacobi homomorphism. This gives other defining equations for the varieties of Weierstrass points $\mathcal{W}_k^r$ defined by Lax.

Introduction

The Jacobi homomorphism φ of a compact Riemann surface M of genus g into its Jacobian variety $J(M)$ depends on its base point, i.e., lower limit of the integral of first kind. Lewittes [10] emphasized the importance of the role of the base point, and showed that a point B is a Weierstrass point if and only if $\theta(W_1 - K) \equiv 0$ with the base point at B, where θ is the Riemann theta function, and $\theta(W) \equiv 0$ means that $\theta(w) = 0$ for all $w \in W$.

The main purpose of this paper is to extend this Lewittes' result. Our result is: The gap sequence at a point B is completely determined by whether $\theta(W_s - W_t - K) \equiv 0$ or $\not\equiv 0$ with the base point at B for any integers s and t.

Although this is merely a version of the Riemann–Roch theorem, it seems to give a new approach to the problem of finding the number of moduli of the space which is formed by all the Riemann surfaces having a Weierstrass point with a specified Weierstrass sequence. Such kinds of problems have been investigated by many mathematicians: Hensel–Landsberg [6], Rauch [14], Farkas [2], Lax [8, 9], Pinkham [13], Rim–Vitulli [15]. We in fact show that if $\mathcal{W}_k^r$ denotes the closed analytic subspace of V defined by Lax, then $\pi(\mathcal{W}_k^r)$ is also

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defined by the condition $\theta(W_r - W_{g-1+r-k} - K) \equiv 0$ $(k \leqq g)$ or $\theta(W_{k-g+r} - W_{r-1} - K) \equiv 0$ $(g \leqq k)$, where $\pi : V \to T_g$ denotes the universal curve over the Teichmüller space T_g. At last we give a defining equation for $\mathcal{W}_g = \mathcal{W}_g^1$.

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§1. Preliminaries

Let M denote a compact Riemann surface of genus $g \geqq 2$. Let D be a divisor on M. We denote by $l(D)$ the dimension of the space of meromorphic functions on M whose divisors are multiples of $-D$, $i(D)$ the dimension of the space of abelian differentials on M whose divisors are multiples of D.

To each point P on M, there corresponds a sequence of g integers n_i, $0 < n_1 < n_2 < \cdots < n_g < 2g$ such that there exists no meromorphic function with a pole of order n_i at P as its only singularity. The sequence is called the gap sequence at B, and the integer n_i is called a gap at P, and their complement in the positive integers are called the nongaps at P. A point with $n_g > g$ is called a Weierstrass point. Thus there exists a meromorphic function on M with a pole of order $m \leqq g$ at a Weierstrass point as its only singularity. There is a finite number of Weierstrass points on M. For more details we refer to [4].

Let $J(M)$ be the Jacobian variety of M, and φ be the Jacobi homomorphism defined by

$$\varphi(D) = (u_1(D), u_2(D), \ldots, u_g(D)) = \left(\int_{rB}^{D} du_1, \int_{rB}^{D} du_2, \ldots, \int_{rB}^{D} du_g \right),$$

where u_i are a basis of abelian differentials of 1st kind and the degree of the divisor D is r. For $r \geqq 0$, let $W_r \subseteq J(M)$ denote the set $\{\varphi(D) \mid D : \text{a positive divisor of degree } r \text{ on } M\}$, which are irreducible subvarieties of dimension r, and let $W_r^s \subseteq W_r$ denote the set $\{\varphi(D) \mid D : \text{a positive divisor of degree } r \text{ on } M \text{ such that } l(D) \geqq s\}$. These subsets W_r^s are analytic subvarieties of $J(M)$, but many not be irreducible.

If S and T are subvarieties of $J(M)$, we can construct the following subvarieties.

$$-S = \{-s \mid s \in S\},$$

$$S + T = \{s + t \mid s \in S, t \in T\},$$

$$S - T = S + (-T),$$

$$S \ominus T = \{u \mid u \in J(M), T + u \subseteq S\} = \bigcap_{t \in T} (S - t).$$

For the proofs of the following lemmas, we refer to [4], [5], [7], [11].

LEMMA 1. $W_r^s \ominus (-W_t) = W_{r+t}^{s+t}$ $(1 \leqq s \leqq r+1)$.

LEMMA 2. $W_r^s \ominus W_t = W_{r-t}^s$ $(t \leqq r-s+1)$.

To prove Lemma 4 (and Lemma 2), we need the next lemma.

LEMMA 3. *If A, B and C are subvarieties of $J(M)$, then*

$$(A \ominus B) \ominus C = A \ominus (B + C).$$

LEMMA 4. $W_r^\nu \ominus (W_s - W_t) = W_{r-s+t}^{\nu+t}$ $(1 \leqq \nu \leqq r-s+1)$.

PROOF. $W_r^\nu \ominus (W_s - W_t) = (W_r^\nu \ominus W_s) \ominus (-W_t) = W_{r-s}^\nu \ominus (-W_t) = W_{r-s+t}^{\nu+t}$.

We denote by θ the Riemann theta function ([4], [10], [12]). The function θ may be considered as a multiplicative function on $J(M)$. The zeros are well-defined, and the zero set is known to be $W_{g-1} + K$, where K is the vector of Riemann constants (Theorem 1). Lewittes [10] studied how the Jacobi homomorphism depends on the base point. We denote φ_B, K_B when B is the base point.

LEMMA 5. (1) $\varphi_{B'}(P) = \varphi_B(P) - \varphi_B(B')$,
(2) $K_{B'} = K_B + (g-1) \cdot \varphi_B(B')$.

It follows also from this lemma that $W_{g-1} + K$ does not depend on the base point. The vectors of Riemann constants form an analytic curve $\tilde{K}$ in $J(M)$ which is independent of the base pont. In fact,

$$\tilde{K} = \bigcup_{P \in M} K_P = K_B + (g-1) \cdot \varphi_B(P),$$

and the right-hand side expression implies the independence on the base point. The curve $\tilde{K}$ can thus be written as $(g-1) \cdot W_1 + K$, and is, of course, contained in $W_{g-1} + K$.

§2. Main result

We now study the relation between the gap sequences of Weierstrass points and the vanishing of the theta function on the subvarieties $W_s - W_t - K$ of the Jacobian variety $J(M)$.

THEOREM 1. (Riemann) *The zero set of the theta function is $W_{g-1} + K$.*

For the proof of this theorem, we refer to [4], [10].

PROPOSITION 1. *Let $e \in J(M)$. If for each positive integer s, $t(s)$ is the integer such that $\theta(W_s - W_{t(s)} - e) \equiv 0$, $\theta(W_s - W_{t(s)+1} - e) \not\equiv 0$, then $t(s)$ is a monotone decreasing function in s, and*

$$e \in (W^{s+1}_{g-1+s-t(s)} \backslash W^{s+1}_{g-2+s-t(s)}) + K,$$

where $A \backslash B$ means the complement of B in A.

PROOF.

$$\theta(W_s - W_{t(s)} - e) \equiv 0 \text{ if and only if}$$

$$W_s - W_{t(s)} - e \subseteq W_{g-1} + K = -(W_{g-1} + K) \text{ if and only if}$$

$$e \in W_{g-1} \ominus (W_{t(s)} - W_s) + K = W^{s+1}_{g-1+s-t(s)} + K.$$

where we used the fact that the theta function is an even function. This means that $\theta(W_s - W_{t(s)+1} - e) \not\equiv 0$ if and only if $e \notin W^{s+1}_{g-2+s-t(s)} + K$.

If $t(s) > t(s-1)$, then

$$W_{s-1} - W_{t(s-1)+1} - e \subseteq W_s - W_{t(s)} - e,$$

which contradicts the hypotheses that $\theta(W_s - W_{t(s)} - e) \equiv 0$ and $\theta(W_{s-1} - W_{t(s-1)+1} - e) \not\equiv 0$. This implies that

$$t(s) \leqq t(s-1) \quad \text{and} \quad g - 1 + s - t(s) > g - 1 + (s-1) - t(s-1).$$

PROPOSITION 2. *Let $e \in J(M)$. If for each non-negative integer t, $s(t)$ is the integer such that $\theta(W_{s(t)} - W_t - e) \equiv 0$, $\theta(W_{s(t)+1} - W_t - e) \not\equiv 0$, then $s(t)$ is a monotone decreasing function of t, and*

$$e \in (W^{s(t)+1}_{g-1+s(t)-t} \backslash W^{s(t)+2}_{g+s(t)-t}) + K.$$

COROLLARY 1. *Given a base point $B \in M$, $\theta(W_s - W_t - K) \equiv 0$ if and only if $l(B^{g-1+s-t}) \geqq s+1$.*

PROOF. $\theta(W_s - W_t - K) \equiv 0$ if and only if $0 \in W^{s+1}_{g-1+s-t}$ if and only if $\varphi(B^{g-1+s-t}) \in W^{s+1}_{g-1+s-t}$.

THEOREM 2 (Riemann). *For $e \in J(M)$ such that $\theta(e) = 0$, $\theta(W_s - W_s - e) \equiv 0$ but $\theta(W_{s+1} - W_{s+1} - e) \not\equiv 0$ if and only if $e \in (W^{s+1}_{g-1} \backslash W^{s+2}_{g-1}) + K$.*

PROOF. This follows easily from the proof of Proposition 1.

This result does not depend on the base point by virtue of the symmetricity of the subvarieties $W_s - W_s$.

LEMMA 6. *Let $s \geqq 1$ be an integer, and $B \in M$ be a base point. If $t(s)$ $(0 \leqq t(s) \leqq g-2)$ is the integer such that $\theta(W_s - W_{t(s)} - K) \equiv 0$, but $\theta(W_s - W_{t(s)+1} - K) \not\equiv 0$, then $g-1+s-t(s)$ is a nongap at B.*

PROOF. It follows from Corollary 1 that $l(B^{g-1+s-t(s)}) \geqq s+1$ and $l(B^{g-1+s-t(s)-1}) \leqq s$. This shows that $g-1+s-t(s)$ is a nongap at B.

LEMMA 7. *Let $t \geqq 0$ be an integer, and $B \in M$ be a base point. If $s(t)$ $(0 \leqq s(t) \leqq g-1)$ is the integer such that $\theta(W_{s(t)} - W_t - K) \equiv 0$, but $\theta(W_{s(t)+1} - W_t - K) \not\equiv 0$, then $g+s(t)-t$ is a gap at B.*

PROOF. From Corollary 1, it follows that $l(B^{g-1+s(t)-t}) \geqq s(t)+1$ and $l(B^{g+s(t)-t}) \leqq s(t)+1$. This shows that $g+s(t)-t$ is a gap at B.

THEOREM 3. *Let $B \in M$ be a base point such that $l(B^{g-1}) = s_0 + 1$. For integers t, $0 \leqq t \leqq s_0$, we have s_0+1 such gaps $g+s(t)-t$ at B as are obtained in Lemma 7 and these are all the gaps at B not less than g. If $s_0 \geqq 1$, then for integers s, $1 \leqq s \leqq s_0$, we have s_0 such nongaps $g-1+s-t(s)$ at B as are obtained in Lemma 6 and these are all the nongaps at B not greater than $g-1$. Thus the gap sequence at B is completely determined.*

PROOF. We prove the second assertion. Riemann's theorem proves that $t(s) \geqq s_0 \geqq s$ so that $g-1+s-t(s) \leqq g-1$. Proposition 1 and Lemma 6 give s_0 different nongaps $g-1+s-t(s) \leqq g-1$ $(1 \leqq s \leqq s_0)$ at B, and these are all the nongaps $\leqq g-1$ at B, since $l(B^{g-1}) = s_0+1$.
The first assertion can be proved similarly.

We can state the analogue of Riemann's vanishing theorem.

PROPOSITION 3. *If $\theta(W_s - W_t - e) \equiv 0$ for $1 \leqq s \leqq t$, then*

$$\frac{\partial^{l_1+\cdots+l_g}}{\partial u_1^{l_1}\cdots \partial u_g^{l_g}} \theta(W_{t-s}+e) \equiv 0, \qquad l_1+\cdots+l_g \leqq s.$$

PROPOSITION 4. *If $\theta(W_s - W_{t+1} - e) \not\equiv 0$ for $1 \leqq s \leqq t$, then there exist indices $i_1, \ldots, i_s$ such that*

$$\frac{\partial^s}{\partial u_{i_1}\cdots \partial u_{i_s}} \theta(W_{t-s+1}+e) \not\equiv 0.$$

PROOFS OF PROPOSITIONS 3 AND 4. We only need to follow the routine argument. For details we refer to Farkas–Kra ([4], pp. 295–297).

§3. Comments

The conditions of Lemmas 6 and 7 imply that W_r^s contains 0 or not with the base point at B. It is stated in Gunning's lecture notes ([5], p. 52) that

$$r \cdot W_1 \cap W_r^s = \{r \cdot \varphi(P) \mid \text{there are at least } s-1 \text{ nongaps at } P \text{ in } (1,2,\dots,r)\}.$$

If 0 is contained in W_r^s with the base point at B, then $\varphi(B') \in W_r^s$. This explains why we can get the information about the gap sequence at the base point.

§4. In relation to the moduli problem

The description of the Weierstrass gap sequence by means of the Riemann theta function in Theorem 2 is closely related to the works by Duma [1], Lax [8, 9]. Lax defined the closed analytic subspaces $\mathcal{W}_k^r$ of V such that

(i) if $k \leqq g$, then

$$|\mathcal{W}_k^r| = \{(s,P): \text{in the gap sequence at } P \in V_s, \text{ there are at least } r \text{ nongaps } \leqq k\},$$

and

(ii) if $k \geqq g$, then

$$|\mathcal{W}_k^r| = \{(s,P): \text{in the gap sequence at } P \in V_s, \text{ there are at least } r \text{ gaps } > k\}.$$

where V_s is the fiber over $s \in T_g$, and proved

THEOREM 4. (Lax) *$\mathcal{W}_k^1 - \mathcal{W}_k^2$ is smooth of pure dimension $k+2g-3$ $(2 \leqq k \leqq g)$ and $\mathcal{W}_k^1 - \mathcal{W}_k^2$, if nonempty, is smooth of pure dimension $4g-k-3$ $(g \leqq k)$.*

Meanwhile, it follows from Corollary 1 that $\pi(\mathcal{W}_k^r)$ is also defined by the condition $\theta(W_r - W_{g-1+r-k} - K) \equiv 0$ $(k \leqq g)$ or $\theta(W_{k-g+r} - W_{r-1} - K) \equiv 0$ $(k \geqq g)$. Thus it is hopeful that the description could be a new approach to the problem of the moduli of Riemann surfaces of genus g having a Weierstrass point with a specified gap sequence. We shall give a more explicit form to the condition $\theta(W_1 - K) \equiv 0$ for $\mathcal{W}_g = \mathcal{W}_g^1$.

Expanding in Taylor series $\theta(W_{g-1} + K) \equiv 0$ about B^{g-1} and $\theta(W_1 - K) \equiv 0$ about B in the case $g = 4$, we obtain that

$$\sum_i \frac{\partial \theta}{\partial u_i}(K) u_i^{(4)}(B) = 0$$

if and only if $\theta(\varphi(t) - K) \equiv 0$, i.e., B is a Weierstrass point.

We can generalize this fact to the next Proposition.

PROPOSITION 5. *A point $B \in M$ is a Weierstrass point if and only if*

$$\sum_{i=1}^{g} \frac{\partial\theta}{\partial u_i}(K)u_i^{(g)}(B) = 0,$$

where B is the base point of the Jacobi homomorphism.

PROOF. We first note that a point B is a Weierstrass point if and only if

$$\begin{pmatrix} u_1'(B) \cdots u_g'(B) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ u_1^{(g)}(B) \cdots u_g^{(g)}(B) \end{pmatrix} \begin{pmatrix} \frac{\partial\theta}{\partial u_1}(K) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial\theta}{\partial u_g}(K) \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

If the rank of the $g \times g$ matrix is less than g, then B is a Weierstrass point ([3], p. 82). If the rank is g, then it follows from the above equation that

$$\frac{\partial\theta}{\partial u_1}(K) = \cdots = \frac{\partial\theta}{\partial u_g}(K) = 0.$$

Then $K \in W_{g-1}^2 + K$, i.e., $\varphi(B^{g-1}) \in W_{g-1}^2$, and B is a Weierstrass point.

NOTE. The idea of exploiting $\theta(W_{g-1} + K) \equiv 0$ at special points has appeared in [3], [4, pp. 313–314]. The equation can also be obtained by using the fact that if $e \in (W_{g-1} \backslash W_{g-1}^2) + K$, then $\Sigma_{i=1}^{g} (\partial\theta/\partial u_i)(e)du_i$ is a holomorphic differential which vanishes at $P_1, \ldots, P_{g-1}$, where $e = \varphi(P_1 \cdots P_{g-1}) + K$.

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